Parametric Surfaces

A surface in $\mathbb{R}^3$ can be described by parametric equations corresponding to vector equation $\vec{s}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$.

Ex: The sphere of radius $r>0$ has parametric equations

$$\vec{s}(\theta,\varphi) = \langle r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi) \rangle \quad \text{for} \quad (\theta,\varphi) \in [0,2\pi] \times [0,\pi].$$

Ex: The torus with inner and outer radii $0 < r < R$ is

$$\vec{s}(\theta,\varphi) = \langle (R + r\sin(\theta))\cos(\varphi), (R + r\sin(\theta))\sin(\varphi), r\cos(\theta) \rangle$$

for $(\theta,\varphi) \in [0,2\pi] \times [0,2\pi]$

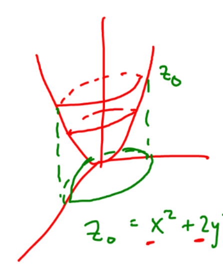
Ex: Let $f:[a,b] \to \mathbb{R}$ be a function. The surface of revolution obtained by revolving $y = f(x)$ about the x-axis has parametrization

$$\vec{s}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta) \rangle \quad \text{for} \quad (x,\theta) \in [a,b] \times [0,2\pi].$$

Ex: Parametrize the elliptic paraboloid $z = x^2 + 2y^2$.

Sol 1: $\vec{s}(u,v) = \langle u, v, u^2 + 2v^2 \rangle$ 1/4

Sol 2: $\vec{s}(r,\theta) = \langle \sqrt{2}\, r\cos(\theta), r\sin(\theta), 2r^2 \rangle$



$x = \sqrt{2}\, r\cos(\theta)$

$y = r\sin(\theta)$

$z = x^2 + 2y^2 = (\sqrt{2}\, r\cos(\theta))^2 + 2(r\sin(\theta))^2 = 2r^2$ 1/1

A surface S is <u>orientable</u> when it does not contain a <u>Möbius band</u>, i.e. when it admits a consistent <u>orientation</u>. Essentially, this means we can choose a consistent normal vector at all points of S.

See GEOGEBRA for an interactive explanation of non orientability.

Note that a smooth parametrization $\vec{s}$ comes equipped with two choices of normal:

$$\pm \frac{\vec{S}_u(u,v) \times \vec{S}_v(u,v)}{|\vec{S}_u(u,v) \times \vec{S}_v(u,v)|}.$$

$\langle x_u, y_u, z_u \rangle$ $\langle x_v, y_v, z_v \rangle$

Note that most of what we do here assumes we are working with a smoothly parametrized orientable surface.

Non-orientable surfaces are fascinating, but understanding them takes some work. We will be content to <u>know they</u> exist, but we won't do much with them...

If you want to learn more about this topic, send me an email, and I'll point you in the right direction to learn more. :)

Tangent Planes

Suppose $\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ parametrizes a surface S smoothly. To compute the tangent plane to S at a point (x_0, y_0, z_0) on S, we must compute a normal vector to the desired plane at (x_0, y_0, z_0).

We can compute this normal vector $\vec{n}$ as the cross product of two non-parallel vectors in the tangent plane. Because $\vec{S}$ is a smooth parametrization, we can use two natural curves in the surface, as follows:

→ Compute an input (u_0, v_0) with $\vec{S}(u_0, v_0) = \langle x_0, y_0, z_0 \rangle$

→ Let $\vec{\alpha}(u) := \vec{S}(u, v_0)$ and $\vec{\beta}(v) := \vec{S}(u_0, v)$. The tangent vectors $\vec{\alpha}\,'(u_0)$ and $\vec{\beta}\,'(v_0)$ lie in the desired plane.

→ Compute $\vec{n} = \vec{\alpha}\,'(u_0) \times \vec{\beta}\,'(v_0)$.

SEE GEOGEBRA

The tangent plane has equation $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$. Alternatively:

$$\left(\vec{S}_u(u_0,v_0) \times \vec{S}_v(u_0,v_0)\right) \cdot \left(\vec{x} - \vec{S}(u_0,v_0)\right) = 0.$$

where $\vec{S}_u(u_0,v_0) = \langle x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0) \rangle$ and

$\vec{S}_v(u_0,v_0) = \langle x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0) \rangle$.

Ex: Compute the tangent plane to the surface $\vec{S}(u,v) = \langle u^2, v^2, u+2v \rangle$ at $(1,1,3)$.

Sol: $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$ w/ $\vec{n} = \vec{S}_u(u_0,v_0) \times \vec{S}_v(u_0,v_0)$

① Compute (u_0,v_0):

$$\vec{S}(u,v) = \langle 1,1,3 \rangle \iff \begin{cases} u^2 = 1 \\ v^2 = 1 \\ u+2v = 3 \end{cases} \iff \begin{cases} u = \pm 1 \\ v = \pm 1 \\ u+2v = 3 \end{cases}$$

Four candidates: $\boxed{(1,1)}$, ~~$(1,-1)$~~, ~~$(-1,1)$~~, ~~$(-1,-1)$~~

$\vec{S}(1,1) = 1+2 = 3$, $\vec{S}(1,-1) = 1-2 = -1$

$\vec{S}(-1,1) = -1+2 = 1$, $\vec{S}(-1,-1) = -1-2 = -3$

② Compute $\vec{S}_u$ and $\vec{S}_v$: $\vec{S}(u,v) = \langle u^2, v^2, u+2v \rangle$

$$\vec{S}_u = \langle x_u, y_u, z_u \rangle = \langle 2u, 0, 1 \rangle$$

$$\vec{S}_v = \langle x_v, y_v, z_v \rangle = \langle 0, 2v, 2 \rangle$$

③ Compute $\vec{S}_u \times \vec{S}_v$:

$$\vec{S}_u \times \vec{S}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = \left\langle \begin{vmatrix} 0 & 1 \\ 2v & 2 \end{vmatrix}, -\begin{vmatrix} 2u & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} \right\rangle = \langle -2v, -4u, 4uv \rangle.$$

④ Write the equation: $(\vec{S}_u \times \vec{S}_v)(1,1) = \langle -2,-4,4 \rangle = -2\langle 1,2,-2 \rangle$

$\therefore$ The tangent plane has equation

$$1(x-1) + 2(y-1) - 2(z-3) = 0 \quad \text{OR} \quad x + 2y - 2z = -3$$

$\boxed{4}$

Ex: Compute an equation of a tangent plane to the surface

$$\vec{s}(u,v) = \langle u+v,\ 3u^2,\ u-v\rangle \quad \text{at} \quad (2,3,0).$$

Sol: $\vec{S}_u \times \vec{S}_v = \vec{n}$, $\quad \vec{s}(u_0,v_0) = \langle 2,3,0\rangle$

$$\vec{S}(u,v) = \langle 2,3,0\rangle \iff \begin{cases} u+v = 2 \\ 3u^2 = 3 \\ u-v = 0 \end{cases} \iff \begin{cases} u = v \\ 2u = 2 \\ u^2 = 1 \end{cases} \iff \begin{cases} u = 1 \\ v = 1 \\ 1^2 = 1 \ \checkmark \end{cases}$$

$$\vec{S}_u = \langle 1,\ 6u,\ 1\rangle \quad \text{and} \quad \vec{S}_v = \langle 1,\ 0,\ -1\rangle$$

$$\therefore \quad \vec{S}_u \times \vec{S}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6u & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} 6u & 1 \\ 0 & -1 \end{vmatrix},\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix},\ \begin{vmatrix} 1 & 6u \\ 1 & 0 \end{vmatrix} \right\rangle$$

$$= \langle -6u,\ -(-1-1),\ -6u\rangle$$

$$= \langle -6u,\ 2,\ -6u\rangle$$

$\therefore$ at $(u,v) = (1,1)$ we have $\vec{n} = \langle -6,2,-6\rangle = -2\langle 3,-1,3\rangle$.

Hence the desired tangent plane has equation:

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

i.e. $\langle 3,-1,3\rangle \cdot \langle x-2,\ y-3,\ z-0\rangle = 0$

i.e. $3(x-2) - (y-3) + 3(z) = 0$

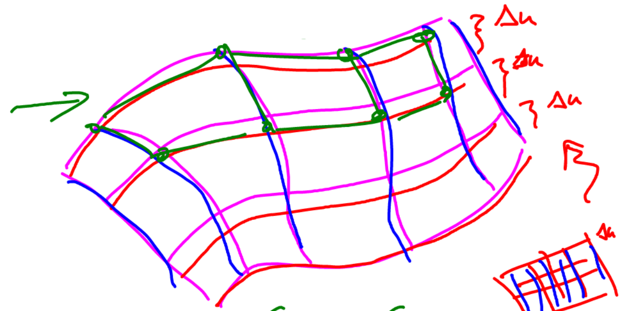
i.e. $3x - y + 3z = 3$

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Surface Integrals

Goal: Integrate function $f: R \subseteq \mathbb{R}^3 \to \mathbb{R}$ over a surface $S \subseteq R$.

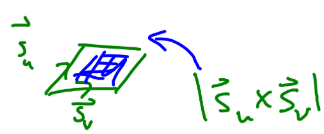
Idea: Let S be a surface with smooth parametrization $\vec{S}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. Then locally S is well-approximated by its tangent planes. In particular, we can approximate S piecewise by polygons.



SEE GEOGEBRA...

Having done so, we then apply our function and add the result scaled by parallelogram size. We thus obtain:

$$\int_S f\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\vec{S}(P_{ij}))\, \Delta S_{ij}$$



where P_{ij} is a point on the (i,j)-polygon and ΔS_{ij} is the area on this polygon. On the other hand, the (i,j) polygon can be approximated by the parallelogram given by the tangent vectors $\vec{S}_u$ and $\vec{S}_v$ at each of these points. Hence the area is

$$\Delta S_{ij} \approx |\vec{S}_u(P_{ij}) \times \vec{S}_v(P_{ij})|\, \Delta u\, \Delta v.$$

Hence, in the limit we obtain the formula:

$$\int_S f\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\vec{S}(P_{ij}))\, |\vec{S}_u(P_{ij}) \times \vec{S}_v(P_{ij})|\, \Delta u\, \Delta v$$

$$= \iint_D f(\vec{S}(u,v))\, |\vec{S}_u(u,v) \times \vec{S}_v(u,v)|\, dA$$

Ex: Compute $\iint_S (x+y+z)\, dS$ where S is the surface parametrized by $\vec{s}(u,v) = \langle u+v,\ u-v,\ 1+2u+v \rangle$ on $(u,v) \in [0,2]\times[0,1]$.

Sol: $$\iint_S f\, dS := \iint_D f(\vec{s}(u,v))\, |\vec{s}_u(u,v) \times \vec{s}_v(u,v)|\, dA$$

$$\vec{s}_u = \langle 1, 1, 2 \rangle \qquad \vec{s}_v = \langle 1, -1, 1 \rangle$$

$$\vec{s}_u \times \vec{s}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right\rangle = \langle 3, 1, -2 \rangle$$

$$|\vec{s}_u \times \vec{s}_v| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$f(\vec{s}(u,v)) = (u+v) + (u-v) + (1+2u+v) = 1 + 4u + v$$

$$\therefore \iint_S f\, dS = \iint_D f(\vec{s}(u,v))\, |\vec{s}_u(u,v) \times \vec{s}_v(u,v)|\, dA$$

$$= \iint_D (1+4u+v)\sqrt{14}\, dA$$

$$= \sqrt{14} \int_{u=0}^{2} \int_{v=0}^{1} (1+4u+v)\, dv\, du$$

$$= \sqrt{14} \int_{u=0}^{2} \left[v + 4uv + \tfrac{1}{2}v^2 \right]_{v=0}^{1} du$$

$$= \sqrt{14} \int_{u=0}^{2} \left(1 + 4u + \tfrac{1}{2}\right) du = \sqrt{14} \int_{u=0}^{2} \left(\tfrac{3}{2} + 4u\right) du$$

$$= \sqrt{14} \left[\tfrac{3}{2}u + 2u^2 \right]_{u=0}^{2}$$

$$= \sqrt{14}\left(\tfrac{3}{2}\cdot 2 + 2\cdot 2^2 - 0 \right) = 11\sqrt{14}$$

//

Ex: Compute $\iint_S y \, dS$ for S the surface parametrized by $\vec{S}(u,v) = \langle u\cos(v), u\sin(v), v \rangle$ for $(u,v) \in [0,1] \times [0,\pi]$.

Sol: $$\iint_S f \, dS = \iint_D f(\vec{S}(u,v)) \, |\vec{S}_u(u,v) \times \vec{S}_v(u,v)| \, dA$$

$$f(\vec{S}(u,v)) = u\sin(v)$$

$$\vec{S}_u = \langle \cos(v), \sin(v), 0 \rangle \qquad \vec{S}_v = \langle -u\sin(v), u\cos(v), 1 \rangle$$

$$\vec{S}_u \times \vec{S}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} \sin(v) & 0 \\ u\cos(v) & 1 \end{vmatrix}, -\begin{vmatrix} \cos(v) & 0 \\ -u\sin(v) & 1 \end{vmatrix}, \begin{vmatrix} \cos(v) & \sin(v) \\ -u\sin(v) & u\cos(v) \end{vmatrix} \right\rangle$$

$$= \langle \sin(v), -\cos(v), u\cos^2(v) + u\sin^2(v) \rangle$$

$$= \langle \sin(v), -\cos(v), u \rangle$$

$$|\vec{S}_u \times \vec{S}_v| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1+u^2}$$

$$D = \{(u,v) : 0 \le u \le 1, \ 0 \le v \le \pi\}$$

$$\therefore \iint_S f \, dS = \iint_D f(\vec{S}(u,v)) \, |\vec{S}_u(u,v) \times \vec{S}_v(u,v)| \, dA$$

$$= \iint_D u\sin(v)\sqrt{1+u^2} \, dA$$

$$= \int_{u=0}^{1} \int_{v=0}^{\pi} u\sin(v)\sqrt{1+u^2} \, dv \, du$$

$$= \int_{u=0}^{1} u\sqrt{1+u^2}\,\left[-\cos(v)\right]_{v=0}^{\pi}\,du$$

$$= \int_{u=0}^{1} u\sqrt{1+u^2}\,\left(-\cos(\pi) - -\cos(0)\right)\,du$$

$$= (1+1)\int_{u=0}^{1} u\sqrt{1+u^2}\,du \qquad \left(\begin{array}{l} w = 1+u^2 \\ dw = 2u\,du \end{array}\right.$$

$$= 2\int_{u=0}^{1} \tfrac{1}{2}(1+u^2)^{1/2}\;2u\,du$$

$$= \int_{u=0}^{1} w^{1/2}\,dw$$

$$= \left[\tfrac{2}{3}\,w^{3/2}\right]_{u=0}^{1}$$

$$= \tfrac{2}{3}\left[(1+u^2)^{3/2}\right]_{u=0}^{1}$$

$$= \tfrac{2}{3}\left((1+1)^{3/2} - (1+0)^{3/2}\right)$$

$$= \tfrac{2}{3}\left(2^{3/2} - 1\right)$$

$$= \tfrac{2}{3}\left(2\sqrt{2} - 1\right) \qquad \boxed{///}$$

Surface Area

We have surface area $A = \iint_S dS = \iint_D |\vec{S}_u \times \vec{S}_v| \, dA$

Where surface S is smoothly parametrized by $\vec{S}(u,v)$.

Ex: Compute the surface area of the surface parametrized by $\vec{S}(u,v) = \langle u^2, uv, \frac{1}{2}v^2 \rangle$ for $(u,v) \in [0,1] \times [0,2]$.

Sol: $A = \iint_S dS = \iint_D |S_u(u,v) \times S_v(u,v)| \, dA$

$\vec{S}_u = \langle 2u, v, 0 \rangle \qquad \vec{S}_v = \langle 0, u, v \rangle$

$$\therefore \vec{S}_u \times \vec{S}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \left\langle \begin{vmatrix} v & 0 \\ u & v \end{vmatrix}, -\begin{vmatrix} 2u & 0 \\ 0 & v \end{vmatrix}, \begin{vmatrix} 2u & v \\ 0 & u \end{vmatrix} \right\rangle$$

$$= \langle v^2, -2uv, 2u^2 \rangle$$

$$\therefore |\vec{S}_u \times \vec{S}_v| = \sqrt{(v^2)^2 + 2(2u^2)v^2 + (2u^2)^2}$$

$$= \sqrt{(v^2 + 2u^2)^2} = v^2 + 2u^2$$

$$\therefore A = \iint_S dS = \iint_D |\vec{S}_u(u,v) \times \vec{S}_v(u,v)| \, dA$$

$$= \int_{u=0}^{1} \int_{v=0}^{2} (v^2 + 2u^2) \, dv \, du = \int_{u=0}^{1} \left[\frac{1}{3}v^3 + 2u^2 v \right]_{v=0}^{2} du$$

$$= \int_{u=0}^{1} \left(\frac{8}{3} + 4u^2 \right) du = \left[\frac{8}{3}u + \frac{4}{3}u^3 \right]_{u=0}^{1}$$

$$= \frac{4}{3}(2 \cdot 1 + 1^3 - 0) = \frac{4}{3} \cdot 3 = 4$$

▨

Ex: Compute the surface area of the sphere of radius r.

Sol:
$$\vec{S}(\theta,\varphi) = \langle r\sin(\varphi)\cos(\theta),\ r\sin(\varphi)\sin(\theta),\ r\cos(\varphi)\rangle$$
on $D = \{(\theta,\varphi) : 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

$$\vec{S}_\theta = \langle -r\sin(\varphi)\sin(\theta),\ r\sin(\varphi)\cos(\theta),\ 0\rangle$$

$$\vec{S}_\varphi = \langle r\cos(\varphi)\cos(\theta),\ r\cos(\varphi)\sin(\theta),\ -r\sin(\varphi)\rangle$$

$$\vec{S}_\theta \times \vec{S}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) & 0 \\ r\cos(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) & -r\sin(\varphi) \end{vmatrix}$$

$$= \langle -r^2\sin^2(\varphi)\cos(\theta),\ -r^2\sin^2(\varphi)\sin(\theta),\ -r^2\sin(\varphi)\cos(\varphi)\sin^2(\theta) - r^2\sin(\varphi)\cos(\varphi)\cos^2(\theta)\rangle$$

$$= -r^2\sin(\varphi)\langle \sin(\varphi)\cos(\theta),\ \sin(\varphi)\sin(\theta),\ \cos(\varphi)\rangle$$

$$|\vec{S}_\theta \times \vec{S}_\varphi| = r^2\sin(\varphi)\sqrt{\sin^2(\varphi)\cos^2(\theta) + \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)}$$

$$= r^2\sin(\varphi)\sqrt{\sin^2(\varphi) + \cos^2(\varphi)} = r^2\sin(\varphi).$$

$$\therefore A = \iint_S dS = \iint_D |\vec{S}_\theta \times \vec{S}_\varphi|\, dA$$

$$= \iint_D r^2\sin(\varphi)\, dA = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} r^2\sin(\varphi)\, d\varphi\, d\theta$$

$$= r^2\int_{\theta=0}^{2\pi}\left[-\cos(\varphi)\right]_{\varphi=0}^{\pi} d\theta = (-\,-1+1)r^2\int_{\theta=0}^{2\pi} d\theta$$

$$= 2r^2\left[\theta\right]_{\theta=0}^{2\pi} = 2r^2(2\pi - 0) = 4\pi r^2$$

///

Ex: Compute the surface area of the torus with minor and major radii $0 < r < R$.

Sol: $\vec{S}(\theta, \varphi) = \langle (R + r\sin(\varphi))\cos(\theta), (R + r\sin(\varphi))\sin(\theta), r\cos(\varphi) \rangle$

on $D = \{(\theta, \varphi) : 0 \le \theta \le 2\pi, \ 0 \le \varphi \le 2\pi\}$

$$\vec{S}_\theta = \langle -(R + r\sin(\varphi))\sin(\theta), (R + r\sin(\varphi))\cos(\theta), 0 \rangle$$

$$\vec{S}_\varphi = \langle r\cos(\varphi)\cos(\theta), r\cos(\varphi)\sin(\theta), -r\sin(\varphi) \rangle$$

$$\vec{S}_\theta \times \vec{S}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(R + r\sin(\varphi))\sin(\theta) & (R + r\sin(\varphi))\cos(\theta) & 0 \\ r\cos(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) & -r\sin(\varphi) \end{vmatrix}$$

$$= \Big\langle -r(R + r\sin(\varphi))\cos(\theta)\sin(\varphi), r(R + r\sin(\varphi))\sin(\theta)\sin(\varphi),$$
$$-r(R + r\sin(\varphi))\cos(\varphi)\sin^2(\theta) - r(R + r\sin(\varphi))\cos(\varphi)\cos^2(\theta) \Big\rangle$$

$$= -r(R + r\sin(\varphi)) \langle \cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi)(\sin^2(\theta) + \cos^2(\theta)) \rangle$$

$$= -r(R + r\sin(\varphi)) \langle \cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi) \rangle$$

$$\therefore |\vec{S}_\theta \times \vec{S}_\varphi| = r(R + r\sin(\varphi)) \sqrt{\cos^2(\theta)\sin^2(\varphi) + \sin^2(\theta)\sin^2(\varphi) + \cos^2(\varphi)}$$
$$= r(R + r\sin(\varphi))$$

$$\therefore A = \iint_S dS = \iint_D |\vec{S}_\theta \times \vec{S}_\varphi| \, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} r(R + r\sin(\varphi)) \, d\varphi \, d\theta$$

$$= r \int_{\theta=0}^{2\pi} \left[R\varphi - r\cos(\varphi) \right]_{\varphi=0}^{2\pi} \, d\theta$$

$$= r \int_{\theta=0}^{2\pi} \Big(\big(2\pi R - r\cos(2\pi)\big) - \big(0R - r\cos(0)\big) \Big) \, d\theta$$

$$= r \int_{\theta=0}^{2\pi} \left(2\pi R - r - 0 + r \right) d\theta$$

$$= 2\pi r R \int_{\theta=0}^{2\pi} d\theta$$

$$= 2\pi r R \left[\theta \right]_{\theta=0}^{2\pi}$$

$$= 2\pi r R \left(2\pi - 0 \right)$$

$$= 4\pi^2 r R$$

∎

No Flux Given

← So we'll compute it!

The flux of a vector field $\vec{V}$ on $\mathbb{R}^3$ across smoothly parametrized orientable surface $S = \{\vec{S}(u,v) : (u,v) \in D\}$ is

$$\iint_S \vec{V} \cdot d\vec{S} = \iint_D \vec{V}(\vec{S}(u,v)) \cdot \left(\vec{S}_u(u,v) \times \vec{S}_v(u,v)\right) dA$$

Recall $\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. This shorthand is frequently useful for making the above computation...

Ex: Compute the flux of $\vec{V} = \langle z, y, x \rangle$ across the unit sphere.

Sol:
$$\vec{S}(\theta, \varphi) = \langle \cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi) \rangle$$

on $D = \{(\theta, \varphi) : 0 \le \theta \le 2\pi, \ 0 \le \varphi \le \pi\}$

$$\vec{V}(\vec{S}(\theta, \varphi)) = \langle \cos(\varphi), \sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta) \rangle$$

$$\vec{S}_\theta = \langle -\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), 0 \rangle$$

$$\vec{S}_\varphi = \langle \cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), -\sin(\varphi) \rangle$$

$$\vec{V} \cdot (\vec{S}_\theta \times \vec{S}_\varphi) = \det \begin{bmatrix} \cos(\varphi) & \sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) \\ -\sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & 0 \\ \cos(\theta)\cos(\varphi) & \sin(\theta)\cos(\varphi) & -\sin(\varphi) \end{bmatrix}$$

$$= \cos(\varphi)\begin{vmatrix} \cos(\theta)\sin(\varphi) & 0 \\ \sin(\theta)\cos(\varphi) & -\sin(\varphi)\end{vmatrix} - \sin(\varphi)\sin(\theta)\begin{vmatrix} -\sin(\theta)\sin(\varphi) & 0 \\ \cos(\theta)\cos(\varphi) & -\sin(\varphi)\end{vmatrix}$$

$$+ \sin(\varphi)\cos(\theta)\begin{vmatrix} -\sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) \\ \cos(\theta)\cos(\varphi) & \sin(\theta)\cos(\varphi)\end{vmatrix}$$

$$= -\cos(\varphi)\cos(\theta)\sin^2(\varphi) - \sin^2(\theta)\sin^3(\varphi)$$

$$+ \sin(\varphi)\cos(\theta)\left(-\sin^2(\theta)\sin(\varphi)\cos(\varphi) - \cos^2(\theta)\sin(\varphi)\cos(\varphi)\right)$$

$$= -\sin^2(\varphi)\left(\cos(\varphi)\cos(\theta) + \sin(\varphi)\sin^2(\theta) + \cos(\varphi)\cos(\theta)\right)$$

$$= -\sin^2(\varphi)\left(2\cos(\varphi)\cos(\theta) + \sin(\varphi)\sin^2(\theta)\right)$$

$$\therefore \iint_S \vec{V}\cdot d\vec{S} = \iint_D \vec{V}(\vec{S}(\theta,\varphi))\cdot\left(\vec{S}_\theta(\theta,\varphi)\times\vec{S}_\varphi(\theta,\varphi)\right)dA$$

$$= -\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin^2(\varphi)\left(2\cos(\varphi)\cos(\theta) + \sin(\varphi)\sin^2(\theta)\right)d\theta\, d\varphi$$

$$= -\int_{\varphi=0}^{\pi}\sin^2(\varphi)\int_{\theta=0}^{2\pi}\left(2\cos(\varphi)\cos(\theta) + \sin(\varphi)\tfrac{1}{2}\left(1-\cos(2\theta)\right)\right)d\theta\, d\varphi$$

$$= -\int_{\varphi=0}^{\pi}\sin^2(\varphi)\left[2\cos(\varphi)\sin(\theta) + \tfrac{1}{2}\sin(\varphi)\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\right]_{\theta=0}^{2\pi}d\varphi$$

$$= -\int_{\varphi=0}^{\pi}\sin^2(\varphi)\left(\left(0 + \tfrac{1}{2}\sin(\varphi)(2\pi - 0)\right) - \left(0 + \tfrac{1}{2}\sin(\varphi)(0-0)\right)\right)d\varphi$$

$$= -\int_{\varphi=0}^{\pi}\sin^2(\varphi)\left(\pi\sin(\varphi) - 0\right)d\varphi \qquad \left(\begin{array}{l} u = \cos(\varphi) \\ du = -\sin(\varphi)\,d\varphi\end{array}\right) \quad \begin{array}{l} u(\pi) = -1 \\ u(0) = 1\end{array}$$

$$= \pi\int_{\varphi=0}^{\pi}\left(1-\cos^2(\varphi)\right)\left(-\sin(\varphi)\right)d\varphi = \pi\int_{u=1}^{-1}(1-u^2)\,du$$

$$= \pi\left[\tfrac{1}{3}u^3 - u\right]_{u=-1}^{1} = \pi\left(\left(\tfrac{1}{3}-1\right) - \left(-\tfrac{1}{3} - -1\right)\right) = -\tfrac{4}{3}\pi \qquad \square$$

Ex: Compute the flux of $\vec{v} = \langle y, -x, 2z \rangle$ across the upper hemisphere of the sphere of radius 2 oriented down.

Sol:

$$\iint_S \vec{v} \cdot d\vec{S} = \iint_D \vec{v}(\vec{S}(u,v)) \cdot (\vec{S}_u(u,v) \times \vec{S}_v(u,v))\, dA$$



$$\vec{S}(\theta,\varphi) = \langle 2\cos(\theta)\sin(\varphi),\ 2\sin(\theta)\sin(\varphi),\ 2\cos(\varphi) \rangle$$

on $D = \{ (\theta, \varphi) : 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{2} \}$

$$\vec{S}_\theta = \langle -2\sin(\theta)\sin(\varphi),\ 2\cos(\theta)\sin(\varphi),\ 0 \rangle$$

$$\vec{S}_\varphi = \langle 2\cos(\theta)\cos(\varphi),\ 2\sin(\theta)\cos(\varphi),\ -2\sin(\varphi) \rangle$$

$$\vec{S}_\theta \times \vec{S}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin(\theta)\sin(\varphi) & 2\cos(\theta)\sin(\varphi) & 0 \\ 2\cos(\theta)\cos(\varphi) & 2\sin(\theta)\cos(\varphi) & -2\sin(\varphi) \end{vmatrix}$$

$$= \langle -4\sin^2(\varphi)\cos(\theta),\ -4\sin^2(\varphi)\sin(\theta),\ -4\sin^2(\theta)\cos(\varphi)\sin(\varphi) - 4\cos^2(\theta)\cos(\varphi)\sin(\varphi) \rangle$$

$$= -4\sin(\varphi) \langle \cos(\theta)\sin(\varphi),\ \sin(\theta)\sin(\varphi),\ \cos(\varphi) \rangle$$

Q: Outward or inward?

A: Test point! $(\theta = 0,\ \varphi = \frac{\pi}{4})$

$$\vec{S}_\theta \times \vec{S}_\varphi(0, \tfrac{\pi}{4}) = -4\tfrac{\sqrt{2}}{2} \langle 1 \cdot \tfrac{\sqrt{2}}{2},\ 0,\ \tfrac{\sqrt{2}}{2} \rangle$$

$$= -2\sqrt{2} \langle \tfrac{\sqrt{2}}{2}, 0, \tfrac{\sqrt{2}}{2} \rangle$$

$$= -\langle 2, 0, 2 \rangle$$

Already inward facing normal! :)



$\vec{v} = \langle y, -x, 2z \rangle$

$$\vec{v}(\vec{s}(\theta,\varphi)) = \langle 2\sin(\theta)\sin(\varphi), -2\cos(\theta)\sin(\varphi), 4\cos(\varphi) \rangle$$

$$\therefore\ \vec{v} \cdot (\vec{s}_\theta \times \vec{s}_\varphi) = -4\sin(\varphi)\left(2\sin^2(\varphi)\cos(\theta)\sin(\theta) - 2\sin^2(\varphi)\cos(\theta)\sin(\theta) + 4\cos^2(\varphi)\right)$$

$$= -16\sin(\varphi)\cos^2(\varphi)$$

$$\therefore\ \iint_S \vec{v} \cdot d\vec{s} = \iint_D -16\sin(\varphi)\cos^2(\varphi)\, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} -16\sin(\varphi)\cos^2(\varphi)\, d\varphi\, d\theta \qquad \left(\begin{array}{ll} u = \cos(\varphi) & u(\tfrac{\pi}{2}) = 0 \\ du = -\sin(\varphi)\, d\varphi & u(0) = 1 \end{array} \right.$$

$$= \int_{\theta=0}^{2\pi} \int_{u=1}^{0} 16u^2\, du\, d\theta = \int_{\theta=0}^{2\pi} \frac{16}{3} u^3 \Big|_{u=1}^{0} d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{16}{3}(0-1)\, d\theta = -\frac{16}{3}\left[\theta\right]_{\theta=0}^{2\pi}$$

$$= -\frac{16}{3}(2\pi - 0) = -\frac{32}{3}\pi$$

Ex: Compute the flux of $\vec{v} = ze^{xy}\vec{i} - 3ze^{xy}\vec{j} + xy\vec{k}$ across the parallelogram $\vec{s}(u,v) = \langle u+v, u-v, 1+2u+v \rangle$ on $(u,v) \in [0,2] \times [0,1]$.

Sol: Left to you as practice. :)

Ex: Compute the flux of $\vec{v} = y\vec{j} - z\vec{k}$ across the surface consisting of $y = x^2 + z^2$ on $0 \le y \le 1$ and $x^2 + z^2 \le 1$ on $y = 1$.

Sol: Left to you (HINT: Sum over the surface pieces...)